Pulses with Tilted Fronts

Usually the loci of constant phase and constant intensity of a pulse of radiation are parallel and normal to the direction of propagation. However, in the presence of angular dispersion special types of wave packets are formed in which the intensity fronts are tilted with respect to the phase fronts. These pulses with tilted fronts have interesting properties which are very useful in certain applications of ultrafast spectroscopy.

1. Wave packets in one dimension

- **Superposition of plane waves**

  \[
  E(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \tilde{A}(\omega) e^{-ik(\omega)z} e^{-i\omega t} \tag{1.1}
  \]

  Here \( \tilde{A}(\omega) \) is the amplitude and \( \varphi(\omega) = k(\omega) z \) is the spectral phase.

- **Taylor expansion of \( \varphi(\omega) \)**

  The Taylor expansion of the spectral phase at the central frequency \( \omega_0 \) up to the second order is:

  \[
  \varphi(\omega) = \varphi(\omega_0) + \varphi'(\omega_0)(\omega - \omega_0) + \frac{1}{2} \varphi''(\omega_0)(\omega - \omega_0)^2 + \ldots \tag{1.2}
  \]

- **Group velocity approximation**

  When only first order term in (1.2) is kept, \( E(z, t) \) can be written as a product of a carrier plane wave and an envelope function A:

  \[
  E(z, t) = e^{-i(\omega_0 t - k_0 z)} \cdot A(t - z/v_g) = A(t - T_g) \tag{1.3}
  \]

  \( T_g \) is the group delay time

  \[
  T_g = \varphi'_0 \cdot k'_0 \cdot z \tag{1.4}
  \]

  and

  \[
  1/v_g = k'_0 = \left( \frac{dk}{d\omega} \right)_{\omega=\omega_0} \tag{1.5}
  \]

  is the reciprocal group velocity.

  The envelope function \( A \) in (1.3) is given by

  \[
  A(t - T_g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\Omega \tilde{A}(\Omega) e^{-i\Omega(t - T_g)} \tag{1.6}
  \]
Note that there is a time shift of the envelope $A(t - T_g)$ but no change in shape. In the group velocity approximation the pulse shape is constant as the pulse propagates.

- **Loci of constant phase and constant amplitude**

  The motion of a locus of *constant phase* follows from the condition
  
  $$-\omega_0 t + k_0 z = \text{const}$$  \hspace{1cm} (1.7)

  or
  
  $$z = v_p t + \text{const}$$

  $$v_p = \frac{\omega_0}{k_0}$$ \hspace{1cm} (1.8)

  where $v_p$ is the phase velocity. Thus propagation velocity of the carrier phase is the *phase velocity*.

  The motion of a locus of *constant amplitude* follows from the condition
  
  $$t - T_g = \text{const}$$  \hspace{1cm} (1.9)

  or
  
  $$z = v_g t + \text{const}$$' \hspace{1cm} (1.10)

  Thus propagation velocity of the loci of constant amplitude is the *group velocity*.

- **Group delay dispersion and group velocity dispersion**

  Now we take into account the quadratic term in the expansion of the phase (1.2).

  $$\varphi_0'' = k_0'' z = \frac{d^2T_g}{d\omega}$$ \hspace{1cm} (1.11)

  $\varphi_0'' = dT_g/d\omega$ is called *group delay dispersion* (GDD). The *group velocity dispersion* (GVD) is given by

  $$\frac{dv_g}{d\omega} = -\frac{\varphi_0''}{(\varphi_0'')}$$ \hspace{1cm} (1.12)

  Equation (1.1) with the second order phase term included reads:

  $$E(z,t) = e^{-i(\omega_0 t - k_0 z)} \frac{1}{\sqrt{2\pi}} \int_\Omega \tilde{A}(\Omega) e^{-i\Omega(T_g)} e^{-i\Omega z} e^{-i\omega_0 t} \Omega^2$$  \hspace{1cm} (1.13)
The quadratic phase term in (1.13) leads to a change in the shape of the envelope, in particular it causes some broadening. If the width of the wave packet (“pulse width”) is given in terms of the root-mean square $\Delta t$, we have

$$
(\Delta t)^2 = (\Delta t_0)^2 + (\phi_0'' \Delta \omega)^2 \tag{1.14}
$$

$\Delta t_0$ is minimum pulse width (spectral phase constant), taken as the root mean square (r.m.s.) value, $\Delta \omega$ is the r.m.s. frequency width, and $\phi_0''$ is group delay dispersion.

### 2. Generalization for 3D propagation

- **3D wave packets**

Now we form a packet of plane waves propagating in 3D. We replace $z \rightarrow r$ and $k \rightarrow k$:

$$
E(r, t) = \frac{1}{\sqrt{2\pi}} \int d\omega \tilde{A}(\omega) e^{-ik(\omega)r} e^{-i\omega t} \tag{2.1}
$$

- **Spectral phase**

The spectral phase is given by the dot product

$$
\phi(\omega) = k(\omega) \cdot r \tag{2.2}
$$

- **Taylor expansion of the phase**

The form of the wave packet is very similar to that of the one-dimensional case

$$
E(r, t) = e^{-i(\omega_0 t - k_0' r)} \frac{1}{\sqrt{2\pi}} \int d\Omega \tilde{A}(\Omega) e^{-i\Omega(t - T_0')} e^{-i\phi_0'' \Omega^2} \tag{2.3}
$$

with the group delay now

$$
T_0' = k_0' \cdot r \tag{2.4}
$$

and the group delay dispersion

$$
\phi_0'' = k_0'' \cdot r \tag{2.5}
$$

- **Loci of constant phase amplitude and pulse duration**

*Phase*:

Constant phase refers to the phase of the carrier wave:
\[-\omega_0 t + \mathbf{k}_0 \cdot \mathbf{r} = \text{constant} \quad (2.6)\]

or

\[
\mathbf{r} \cdot \mathbf{k}_0 / k_0 = \text{constant} + \omega t / k_0 = \text{constant} + v_p t
\quad (2.7)
\]

(2.7) is the equation of a plane, the plane of constant carrier phase. The phase planes move with the velocity \(v_p = \omega_0 / k_0\) in the direction given by \(\mathbf{k}_0\).

**Amplitude:**
Constant Amplitude within the envelope is given by the condition

\[
T_g - t = \text{constant}
\quad (2.8)
\]

which can be written

\[
\mathbf{r} \cdot \mathbf{k}'_0 / k'_0 = \text{constant} + t / k'_0 = \text{constant} + v_g t
\quad (2.9)
\]

Thus planes of constant amplitude move with the group velocity in the direction \(\mathbf{k}'_0\).

**Pulse width:**
Constant pulse width is given by the condition that the quadratic phase term be constant:

\[
\varphi''_0 = \mathbf{k}_0 \cdot \mathbf{r} = \text{constant}
\quad (2.10)
\]

Note that this set of planes given by (2.10) does not move. On each plane we would observe the *same pulse within that plane*. However, the observed pulse width depends on \(\mathbf{r} \cdot \mathbf{k}_0''\), the distance of the plane from the origin.

Figure 2.1 illustrates a wave packet with tilted pulse fronts where the different directions of the phase fronts (\(\mathbf{k}_0\)) and the pulse fronts (\(\mathbf{k}'_0\)) are indicated.

![Figure 2.1](image-url)
3. Diffraction gratings

Reflecting diffraction gratings produce angular dispersion of the reflected light. The direction of the reflected wave depends on its frequency. Let us discuss the properties of a diffraction grating with reference to Figure 3.1.

![Figure 3.1](image)

- **Grating equation**

The frequency dependence of the direction of the reflected wave follows from the grating equation

\[
\sin \theta(\omega) = \sin \theta_1 + (2\pi c/\omega) \nu
\]

where \( \nu \) is the diffraction order, \( g \): grating constant, \( \theta(\omega) \): angle of the reflected wave, \( \theta_1 \): angle of the incident wave.

- **Frequency dependence of the wave vector of the reflected wave**

The wave vector of the reflected light with reference to the x-y-coordinate axes (see Figure 3.1) can be written:

\[
\mathbf{k}(\omega) = \frac{\omega}{c} \begin{bmatrix} \cos \theta(\omega) \\ \sin \theta(\omega) \end{bmatrix}
\]

Let us introduce unit vectors parallel and perpendicular to \( \mathbf{k}_0 \):
\[
e_{\parallel} = \begin{bmatrix} \cos \theta_0 \\ \sin \theta_0 \end{bmatrix}, \quad e_{\perp} = \begin{bmatrix} -\sin \theta_0 \\ \cos \theta_0 \end{bmatrix}
\]  

(3.3)

Here \( \theta_0 = \theta(\omega_0) \) is the reflection angle of the carrier wave (center frequency \( \omega_0 \)).

- **First derivative of** \( k(\omega) \)

\[
k' = \frac{1}{c} \left( e_{\parallel} + \omega \frac{d\theta}{d\omega} e_{\perp} \right)
\]

(3.4)

- **Second derivative of** \( k(\omega) \)

\[
k'' = \frac{\omega}{c} \left( -\left( \frac{d\theta}{d\omega} \right)^2 e_{\parallel} + \left( \frac{2}{\omega} \frac{d\theta}{d\omega} + \frac{d^2\theta}{d\omega^2} \right) e_{\perp} \right)
\]

(3.5)

- **Angular dispersion of the grating**

Taking the first derivative with respect to \( \omega \) of the grating equation (3.1) we obtain:

\[
\frac{d\theta}{d\omega} = -\frac{2\pi c}{\omega^2 \cos \theta(\omega)}
\]

(3.6)

- **Angle of pulse front tilt**

Let \( \gamma \) be the angle between the pulse fronts and the phase fronts (Figure 2.1), i.e.:

\[
\gamma = \varepsilon(k_0, k'_0)
\]

(3.7)

With (3.2) and (3.4) we have

\[
\cos \gamma = \frac{k_0 \cdot k'_0}{k_0 \cdot k'_0}, \quad \tan \gamma = \omega \frac{d\theta}{d\omega} \bigg|_{\omega = \omega_0}
\]

(3.8)

With the grating dispersion (3.6) we obtain

\[
\tan \gamma = \frac{\lambda_0}{g \cos \theta_0}
\]

(3.9)

where \( \lambda_0 \) is the center wavelength.

- **Pulse broadening**

To calculate the pulse broadening we need the second derivative of the wave vector as given by (3.5). With the first and second derivative of \( \theta(\omega) \) we obtain:
The factor in front of the bracket is the length of $k''$, $k'' = |k''|$. The vector between the brackets in (3.10) is perpendicular to the grating plane. Thus the planes of constant pulse width are parallel to the grating.

- **Group delay dispersion for a grating**

The group delay dispersion is given by the dot product $k'_0 \cdot r$ (Eq. (2.5))

$$\varphi''_0 = -\frac{\lambda_0^3 d}{2 \pi g^2 c^2 \cos^2 \theta_0}$$

(3.11)

where $d$ is the distance travelled in the direction of $k_0$. For large broadening the pulse width is approximately given by

$$\Delta t = \frac{\varphi''_0}{2 \Delta t_0}$$

(3.12)

where $\Delta t_0$ is the width of the original bandwidth limited pulse.

- **Summary diffraction grating**

1. **Pulse front tilt angle**:

$$\tan \gamma = \left( \frac{\omega \frac{d \theta}{d \omega}}{\omega_0} \right)_{\omega = \omega_0} = \frac{\lambda_0}{g \cos \theta_0}$$

(3.13)

2. **Group delay dispersion**:

The group velocity $\varphi''_0$ dispersion along the direction of propagation ($k_0$) as a function of the distance $d$ from the grating is (3.11):

$$\varphi''_0 = k''_0 \cdot r = -\frac{\lambda_0^3 d}{2 \pi g^2 c^2 \cos^2 \theta_0}$$

(3.14)

3. **Pulse broadening**

A second parallel grating at a distance $d$ from the first stops the pulse broadening. The width of the output pulse is (1.14):

$$(\Delta t)^2 = (\Delta t_0)^2 + (\varphi''_0)^2 (\Delta \omega)^2$$

(3.15)

By reflecting a pulse from a grating a wave packet with tilted amplitude (or intensity) fronts can be generated. However, because of the GDD produced by the grating the wave
packet broadens very rapidly during propagation. The shorter the initial pulse the faster is the broadening.

4. Pair of confocal lenses

It can be seen from (3.14) that the group delay dispersion of a diffraction grating is always negative. However, it is well known that with a combination of lenses and gratings GDD of arbitrary magnitude and sign can be produced. In particular, various types of zero dispersion delay lines can be constructed which produce wave packets with tilted pulse fronts free of GDD, that is, the pulse broadening due to the grating dispersion is fully compensated.

We will now calculate the spectral phase acquired by the individual frequency component of the wave packet when it travels from the grating through a lens. It is useful to distinguish two regions: (1) the space between the grating and the lens, and (2) the space between the lens and the focal plane. In region (1) and (2) the spectral components can be considered as plane waves and as spherical waves, respectively.

- Region (1): Phase of the plane wave

![Figure 4.1](image)

For a monochromatic plane wave the spectral phase is given by (2.2). From Figure 4.1 we obtain the phase change upon travelling a distance $z$ along the propagation

$$\varphi_{\text{PW}}(\omega) = \frac{\omega}{c} z \cos(\Theta(\omega)),$$

where $\Theta(\omega) = \theta(\omega) - \theta_0$.

The second derivative calculated from (4.1) is
\[
\varphi_0^{PW} = -\frac{\omega}{c} z \left( \frac{d\varpi}{d\omega} \right)^2 = -\frac{\omega}{c} z \left( \frac{d\varpi}{d\omega} \right)^2 .
\]

(4.2)

Using (3.6) for the angular dispersion of the grating we obtain the result derived earlier (3.11). This confirms the equivalence of deriving the GDD from (4.1) and (3.10).

- **Region 2: Phase of the spherical wave**

![Figure 4.2](image)

The lens transforms the plane wave into a spherical wave converging to the focal plane. By inspection of Figure 4.2 we find that the phase increment of the spherical wave upon traveling from the lens to the focal plane is

\[
\varphi^{SP}(\omega) = \left( \frac{\omega}{c} \right) f \cos(\Theta(\omega)) ,
\]

(4.3)

where \( f \) is the focal length.

The second derivative \( 0f \) (4.3) is readily calculated to be (mind that \( \Theta(\omega) = \varpi(\omega) - \varpi_0 \))

\[
\varphi_0^{SW} = -\frac{\omega}{c} f \left( \frac{d\varpi}{d\omega} \right)^2 = -\frac{\omega}{c} f \left( \frac{d\varpi}{d\omega} \right)^2 .
\]

(4.4)

Comparison of (4.4) and (4.2) reveals an important result. The second derivative of the phase of the plane wave and the spherical wave have **opposite signs**. In particular, if the grating is placed in the front focal plane, we have \( z = f \) and it follows that

\[
\varphi_0^{PW} + \varphi_0^{SW} = 0 .
\]

(4.5)

Thus the group delay dispersion accumulated during the propagation from the front focal plane to the lens is cancelled during the subsequent travel to the back focal plane. This grating configuration is called **zero dispersion delay line**.
• 4f telescope arrangement

Figure 4.3 shows the so called 4f telescope arrangement which consists of a pair of confocal lenses and two identical gratings. One grating is placed in the front focal plane of the first lens, and the other in the back focal plane of the second lens.

The 4f arrangement is fully symmetric with respect to the confocal plane. We have seen that the phase increment during the propagation on the left of the symmetry plane is zero (4.5). Thus the total GDD over the 4f arrangement is also zero.

After the reflection from the first grating the broadened pulse passes through the confocal lenses. The input and the output signal of the confocal pair are completely identical. The broadened pulse at the exit of the second lens is recompressed during its travel to the second grating. The final reflection from the second grating fully restores the original incident pulse.

The close-up of the output region shown in Figure 4.4 illustrates the situation when we place a target in the back focal plane of the 4f arrangement. The blue line indicates the surface of the grating, that is, the plane of constant pulse duration. Note that only in the center of the target the pulse is fully recompressed. At the upper edge of the beam the pulse has not yet attained the minimum duration. On the other hand, at the lower edge of the beam the target is behind the plane minimum pulse width, that is, the pulse reaching this part of the target is again stretched to some extent.
**Imaging properties of the confocal pair**

The interesting imaging properties of the confocal pair are illustrated in Figure 4.5. The object (G) and the image (B) distances $g_1$ and $b_2$ (and $g_1'$ and $b_2'$) obey the relations

$$g_1 + b_2 = 2f; \quad g_1' + b_2' = 0$$  \hspace{1cm} (4.6)

Using the well-known Gauss imaging formula it can be shown that the magnification is given by

$$M = \frac{b_1}{g_1} \frac{b_2}{g_2} = -1,$$  \hspace{1cm} (4.7)

where $b_1$ is the image distance for the intermediate image produced by the first lens (not shown in Figure 4.5), and $g_2$ is the object distance for the image produced by the second lens. Note that the magnification is independent of the parameters. No matter where the object G is placed, we always obtain a 1:1 inverted image.
Variable group delay dispersion

The discussion of Figure 4.3 has shown that the total group delay dispersion vanishes when the two gratings are placed in the focal planes. In Figure 4.6 we consider a more general configuration where the gratings are placed at arbitrary positions.

![Figure 4.5](image)

To be specific, suppose the first grating is placed at \( d_1 \) in front of the first lens (counting \( d_1 \) positive), and the second at a distance \( d_2 \) behind the second grating (counting \( d_2 \) positive). Using (4.2) and (4.4) the total group delay dispersion \( \varphi_0^T \) after the second grating is given by

\[
\varphi_0^T = \frac{\omega}{c} \left( \frac{d\vartheta}{d\omega} \right)^2 (d_1 + d_2)
\]

(4.8)

The condition for zero dispersion of general, asymmetric zero dispersion delay line is:

\[
d_1 + d_2 = 0
\]

(4.9)

For \( (d_1 + d_2) < 0 \) and \( (d_1 + d_2) > 0 \) group delay dispersion is positive and negative, respectively.

In the example of Figure 4.6 grating 1 is placed between the focal plane and the lens, i.e. \( d_1 \) is negative. The distance of grating 2 is \( d_2 = |d_1| = d_1 \), and thus \( d_1 + d_2 = 0 \). The special case \( d_1 = d_2 = 0 \) is the symmetric zero dispersion delay line shown in Figure 4.3.

Finite beam width

So far we have ignored the finite width of the beams. However, the full recovery of the pulse at the output grating (grating 2) requires that there is spatial overlap of all spectral
components. Since the imaging condition (4.6) and the condition of zero dispersion (4.9) are the same, the required spatial overlap is always assured in a general zero dispersion delay line.

However, if the purpose of the grating arrangement is to generate a stretched pulse with positive or negative chirp \((d_1 + d_2 \neq 0)\) the gratings are not imaged onto each other and there is only partial spectral overlap depending on the width of the beam. An example of such a situation is depicted in Figure 4.7 which shows a configuration corresponding to a pulse stretcher with positive group velocity dispersion.

![Figure 4.7](image)

The red and the blue lines indicate the dispersion of the spectral components of the pulse by the first grating. The red and the blue beam do not overlap on the surface of the second grating. The output is split into parallel, but spatially separated beams. Under these circumstances we do not obtain the desired well defined stretched pulse but instead a signal with a more complicated spatial and temporal behavior. The spatial overlap problem can be avoided, for example, by retro-reflecting the output and passing the beam through the gratings a second time.

5. Example of a pulse tilter

We will discuss now the special case of a pulse tilter to be used for the compensation of the velocity mismatch in a RHEED experiment. As shown in Figure 5.1 the electron beam comes in at grazing incidence. It can be shown that the delay time between the electron and the laser beam is minimized for normal incidence of the laser. For this geometry the required tilt angle \(\gamma\) of the laser pulse is given by
\[ \tan \gamma = \frac{c}{V_{el}} \cos \alpha \approx \frac{c}{V_{el}} \]  \hspace{1cm} (5.1)

where \( V_{el} \) is the electron velocity and \( \alpha \) is the grazing angle of the electron beam. For an electron with a kinetic energy of 30 keV and \( \alpha = 5^\circ \) we obtain \( \gamma = 71.5^\circ \).

Figure 5.1

Table 1 shows the experimental parameter which will be used to calculate the characteristics of the pulse tilter discussed in this section:

<table>
<thead>
<tr>
<th>Wavelength</th>
<th>Grating constant</th>
<th>Beam width</th>
<th>Littrow angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_0 = 800 \text{ nm} )</td>
<td>( g = 0.5 \mu \text{m} ) ( (2000 \text{ l/mm}) )</td>
<td>( \Delta = 2 \text{ mm} )</td>
<td>( \Theta_0 = 53^\circ )</td>
</tr>
</tbody>
</table>

Table 1

- **Grating geometry**

For a grating with 2000 l/mm operated in Littrow configuration (the angle of incidence \( \theta_i \) and the diffraction angle \( \theta_0 \) are opposite: \( \theta_0 = -\theta_i \), see Figure 5.2) the diffraction angle is

\[ \sin \theta_0 = \frac{\lambda}{2g} \approx 53^\circ \]  \hspace{1cm} (5.2)

The tilt angle (3.13) obtained for this grating configuration is
\[ \gamma = \arctan \left( \frac{\lambda_0}{g \cos \theta_0} \right) \approx 69^\circ \]  

(5.3)

The tilt angle can be readily fine-tuned to satisfy (5.1).

![Figure 5.2](image)

- **Residual group delay dispersion**

The residual GDD at the edges of the sample was explained in the discussion of Figure 4.4. From the Figure we find

\[ \Delta s = \frac{1}{2} \Delta \tan \theta_0 \]  

(5.4)

With (3.14) we obtain the GDD at the edges

\[ t_{GDD} = \sqrt{\frac{\lambda_0^3 \Delta \sin \theta_0}{4\pi g^2 c^2 \cos \theta_0}} \]  

(5.5)

The quantity \( t_{GDD} \) is an important parameter that determines the characteristics of the pulse tilter.

- **Pulse broadening**

Let \( t_0 \) and \( t \) be the pulse with at the center and the edges, respectively. The pulse broadening is then given by the formula

\[ t = \sqrt{t_0^2 + (\Phi \Delta \omega)^2} \]  

(5.6)

Let us further assume a chirp-free Gaussian pulse. In this case the time-bandwidth product is given by

\[ \Delta \omega t_0 = \frac{1}{2} \]  

(5.7)
and we simply have $\Delta \omega = 1/(2t_0)$.

The pulse broadening due to the residual dispersion increases with the bandwidth. A very short input pulse will thus lead to substantial broadening. The broadened width $t$ can be written

$$\frac{t}{t_{GDD}} = \sqrt{\left(\frac{t_0}{t_{GDD}}\right)^2 + \frac{1}{4} \left(\frac{t_{GDD}}{t_0}\right)^2}$$

(5.8)

Examination of (5.8) shows that the ratio $t/t_{GDD}$ reaches a minimum value $t/t_{GDD} = 1$ for $t_0/t_{GDD} = 1/\sqrt{2}$. Thus the width $t$ of the broadened Gaussian pulse at the sample edges and the optimum width of the input pulse $t_0$ are

$$t = t_{GDD}; \quad t_0 = t_{GDD}/\sqrt{2}$$

(5.9)

Let us use (5.5) to calculate $t_{GDD}$ for the parameter given in Table 1:
The numerical result is $t_{GDD} = 116$ fs. Thus the optimum input pulse width is $t_0 = 82$ fs. At the edges of the sample this pulse is broaden to $t = t_{GDD} = 116$ fs.

Other cases can be readily calculates using the formulae given above. For example, with a shorter than optimum (bandwidth-limited) pulse $t_0 = 50$ fs we have $t = 143$ fs.